This equivalence is a fundamental property of how rotational transformations compose and relates directly to the difference between fixed-axis (extrinsic) and rotating-axis (intrinsic) rotations. Let's analyze and compare the two sequences.

This property, even though very fundamental, can be quite non-intuitive. I went on a rabbit hole of trying to prove this using rotation matrices and it got quite complicated. However, I found a very nice explanation on StackExchange, which led me to another discussion on Blender StackExchange. And learning this, we personally experienced this inside blender while animating body frame rotations with global frame rotations.

The video we rendered can be found here.

Now that's about visualization, let's see why it becomes reversed mathematically.

Setup

We'll consider two rotations:

- Rotation by angle α about the X-axis
- Rotation by angle γ about the Z-axis

Rotation Matrices

The fundamental rotation matrices are:

$$R_{x(\alpha)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$R_{z(\gamma)} = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Extrinsic Rotations (Fixed World Frame)

In extrinsic rotations, we rotate about fixed world axes:

1. First rotation: $R_{x(\alpha)}$ about world X-axis
2. Second rotation: $R_{z(\gamma)}$ about world Z-axis

The composition is:

We just pre-multiply the matrices in the order of application:

$$R_{\text{extrinsic}} = R_{z(\gamma)} R_{x(\alpha)}$$

Intrinsic Rotations (Body Frame)

In intrinsic rotations, we rotate about the body's current axes:

1. First rotation: $R_{z(\gamma)}$ about body Z-axis
2. Second rotation: about the **new** body X-axis (which has been rotated by $R_{z(\gamma)}$)

To express the second rotation in world coordinates:

- We need to rotate by α about the current body X-axis
- This axis is $R_{z(\gamma)} \hat{x}$, where $\hat{x} = [1, 0, 0]^T$

The rotation about an arbitrary axis $\hat{x}$ by angle θ is given by the Rodrigues formula. For our case:

$$R_{\text{second}} = R_{z(\gamma)} R_{x(\alpha)} R_{z(\gamma)}^T$$

This is because:
1. $R^T_{z(\gamma)}$ rotates the current body frame back to align with world frame
2. $R_{x(\alpha)}$ performs the rotation about the original X-axis
3. $R_{z(\gamma)}$ rotates back to the current body frame

One more intuitive way to think about this is that because you are rotating about the new X-axis but with the fundamental rotation matrix being defined in the original frame, you need to "undo" the previous rotation to align the axis, perform the rotation, and then "redo" the previous rotation.

The complete intrinsic rotation is:

$$R_{\text{intrinsic}} = R_{\text{second}} * R_{\text{first}} = \left[R_{z(\gamma)} R_{x(\alpha)} R^T_{z(\gamma)}\right] * R_{z(\gamma)}$$

Simplifying:

$$R_{\text{intrinsic}} = R_{z(\gamma)} R_{x(\alpha)} \left[R^T_{z(\gamma)} R_{z(\gamma)}\right] = R_{z(\gamma)} R_{x(\alpha)} I = R_{z(\gamma)} R_{x(\alpha)}$$

Comparing Results

We have:

$$R_{\text{extrinsic}} = R_{z(\gamma)} R_{x(\alpha)}$$

$$R_{\text{intrinsic}} = R_{z(\gamma)} R_{x(\alpha)}$$

Therefore:

$$R_{\text{intrinsic}} = R_{\text{extrinsic}}$$

Generalization

This proves that for our specific case:
- Intrinsic rotation: Z then X
- Extrinsic rotation: X then Z

are equivalent. In general:
- Intrinsic rotations in order A then B then C
- Extrinsic rotations in reverse order C then B then A

yield the same final orientation.

Why the Order Reverses

The key insight is that each intrinsic rotation occurs in a frame that has been transformed by all previous rotations. The composition of these transformations mathematically reverses the order compared to extrinsic rotations.

For intrinsic rotations:

$$R_{\text{intrinsic}} = R_A * (R_B * R_C)$$

For extrinsic rotations (reverse order):

$$R_{\text{extrinsic}} = R_C * R_B * R_A$$

These are equivalent because the intrinsic rotations are applied in the local frame, which is equivalent to applying them in reverse order in the fixed frame.