Chapter 4

Gimbal Lock

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4.1 Introduction

To achieve all possible orientations of an object in $\mathbb{R}^3$ there are three degrees of freedom that must be accounted for. For example in aircraft control these are typically called roll, pitch and yaw. Roll is rotation along the front-to-back axis. Pitch is rotation along the side-to-side axis, this corresponds to the plane pointing upwards or downwards. Yaw is rotation along a vertical axis. All of these are necessary for the proper functioning of the aircraft.

4.2 Euler Angles

One way to achieve all possible orientations is to allow rotations in each of the major axes and then combine these rotations to orient our object. It's certainly possible to achieve any orientation this way. This method is nice because it's intuitive.

The matrix rotating about the x-axis by an angle of α equals:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

The matrix rotating about the y-axis by an angle of β equals:

$$R_y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

The matrix rotating about the z-axis by an angle of γ equals:

$$R_z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each of these the direction of rotation follows the right-hand-rule with the thumb pointing in each axis' positive direction.

In order to achieve any orientation of our object we'll probably need all three of these rotations. But how to combine them? Typically an order is chosen and then the result can be used to obtain any orientation we wish.

If pick an order, say z then y then x then the result is:

$R_{xyz}(\alpha, \beta, \gamma) = R_x(\alpha)R_y(\beta)R_z(\gamma) =$

$$\begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha \\ \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta & \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{bmatrix}$$

So now for example if we start with a rocket which is pointing upwards and we want it to point towards the positive x-axis we rotate only about the y-axis by $\pi/2$. We can track the point $[0, 0, 1]^T$ to see:

$$R_{xyz}(0, \pi/2, 0) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let's be even more robust. Suppose we have a rocket whose nose is $[0, 0, 1]^T$ and which has a logo on the side $[0, 1, 0]^T$. and suppose we want the rocket to point along the positive ray $z = x$ with the logo below it. We first rotate about the z-axis by $-\pi/2$ and then about the y-axis by $\pi/4$. We can track the points to see:

$$R_{xyz}(0, \pi/4, -\pi/2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$R_{xyz}(0, \pi/4, -\pi/2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

4.3 Gimbal Lock

Suppose we set $\beta = \pi/2$. The resulting rotation matrix is:

$$R_{xyz}(\alpha, \pi/2, \gamma) = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{bmatrix}$$

If we've oriented our object using this combination then at this instant changing α and γ has the same effect. We have therefore lost a degree of freedom of movement because we only have two degrees, that of β and that of $\alpha + \gamma$.

To elaborate further, our choice of order has ramifications. In the above case if $\beta = \pi/2$ then changing γ doesn't yield a rotation about the z-axis as expected, for example an isolated rotation about the z-axis by $\pi/2$ should take $[0, 1, 0]^T$ to $[-1, 0, 0]^T$ but:

$$R_{xyz}(0, \pi/2, \pi/2) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is because our $R_y(\pi/2)$ messes up the result.

Is it even possible to make a change in any of α, β, γ at ths instant when $\alpha = 0$, $\beta = \pi/2$ and $\gamma = 0$ which would result in a rotation about the z-axis?

Consider an object with two critical points, $[0, 1, 0]^T$ and $[-1, 0, 0]^T$.

Suppose we have set $\alpha = 0$, $\beta = \pi/2$, $\gamma = 0$. Observe that:

$$R_{xyz}(0, \pi/2, 0) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P$$

$$R_{xyz}(0, \pi/2, 0) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = Q$$

Suppose at this instant we want to make changes to the angles so as to effectively rotate P about the z-axis. This means that P needs to move (specifically y needs to decrease while x increases or decreases) while Q needs to remain fixed.

Consider what happens at this instant if we modify α, β and γ by a, b, and c respectively:

$$R_{xyz}(a, \pi/2 + b, c) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin b \sin c \\ \cos a \cos c - \sin a \cos b \sin c \\ \sin a \cos c + \cos a \cos b \sin c \end{bmatrix} = P'$$

and

$$R_{xyz}(a, \pi/2 + b, c) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin b \cos c \\ -\cos a \sin c - \sin a \cos b \cos c \\ \cos a \cos b \cos c - \sin a \sin c \end{bmatrix} = Q'$$

For Q' to remain fixed at $[0, 0, 1]^T$ we need $Q'_1 = \sin b \cos c$ to remain at 0. This requires b to remain at 0. However if b remains at 0 then $P'_1 = \sin b \sin c$ remains at 0 too, and we can't rotate. This is our lock.

4.4 Avoiding Gimbal Lock

The only way to avoid gimbal lock is to take a different approach to describing rotations. Quaternions are one of the most popular ways.