Chapter 9

Rotation About an Arbitrary Axis

9.1 Quick Review

Given a point $P = (x, y, z, 1)$ in homogeneous coordinates, let $P' = (x', y', z', 1)$ be the corresponding point after a rotation around one of the coordinate axis has been applied. You will recall the following from our studies of transformations:

1. Rotation about the x-axis by an angle θ_x, counterclockwise (looking along the x-axis towards the origin). Then $P' = R_x P$ where the rotation matrix, R_x,is given by:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Rotation about the y-axis by an angle θ_y, counterclockwise (looking along the y-axis towards the origin). Then $P' = R_y P$ where the rotation matrix, R_y,is given by:

$$R_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Rotation about the z-axis by an angle θ_z, counterclockwise (looking along the z-axis towards the origin). Then $P' = R_z P$ where the rotation matrix, R_z,is given by:

$$R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that the above transformations also apply to vectors.

You will also recall that

$$\begin{aligned} R_x^{-1} &= R_x^T \\ R_y^{-1} &= R_y^T \\ R_z^{-1} &= R_z^T \end{aligned}$$

This means in particular that these matrices are orthogonal. It can also be proven that the product of two orthogonal matrices is itself an orthogonal matrix (see problems at the end of the chapter). So, if we combine several rotations about the coordinate axis, the matrix of the resulting transformation is itself an orthogonal matrix.

One way of implementing a rotation about an arbitrary axis through the origin is to combine rotations about the z, y, and $x - axes$. The matrix of the resulting transformation, R_{xyz}, is

$$R_{xyz} = R_x R_y R_z = \begin{bmatrix} C_y C_z & -C_y S_z & S_y \\ S_x S_y C_z + C_x S_z & -S_x S_y S_z + C_x C_z & -S_x C_y \\ -C_x S_y C_z + S_x S_z & C_x S_y S_z + S_x C_z & C_x C_y \end{bmatrix} \quad (9.1)$$

where

$$C_i = \cos\theta_i \text{ and } S_i = \sin\theta_i \text{ for } i = x, y, z$$

From what we noticed above, R_{xyz} is an orthogonal matrix. This means that its inverse is its transpose.

9.2 Rotation About an Arbitrary Axis Through the Origin

Goal: Rotate a vector $\mathbf{v} = (x, y, z)$ about a general axis with direction vector $\widehat{r}$ (assume $\widehat{r}$ is a unit vector, if not, normalize it) by an angle θ (see figure 9.1). Because it is clear we are talking about vectors, and vectors only, we will omit the arrow used with vector notation.

We begin by decomposing $\mathbf{v}$ into two components: one parallel to $\widehat{r}$ and one perpendicular to $\widehat{r}$. Let us denote $\mathbf{v}_\parallel$ the component parallel to $\widehat{r}$ and $\mathbf{v}_\perp$ the component perpendicular to $\widehat{r}$. You will recall from our study of vectors that

$$\begin{aligned} \mathbf{v}_\parallel &= comp_{\widehat{r}}\mathbf{v} \\ &= \frac{\mathbf{v} \cdot \widehat{r}}{\|\widehat{r}\|^2}\widehat{r} \\ &= (\mathbf{v} \cdot \widehat{r})\,\widehat{r} \text{ since } \widehat{r} \text{ is a unit vector} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{v}_\perp &= orth_{\widehat{r}}\mathbf{v} \\ &= \mathbf{v} - (\mathbf{v} \cdot \widehat{r})\,\widehat{r} \end{aligned}$$

And we have

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

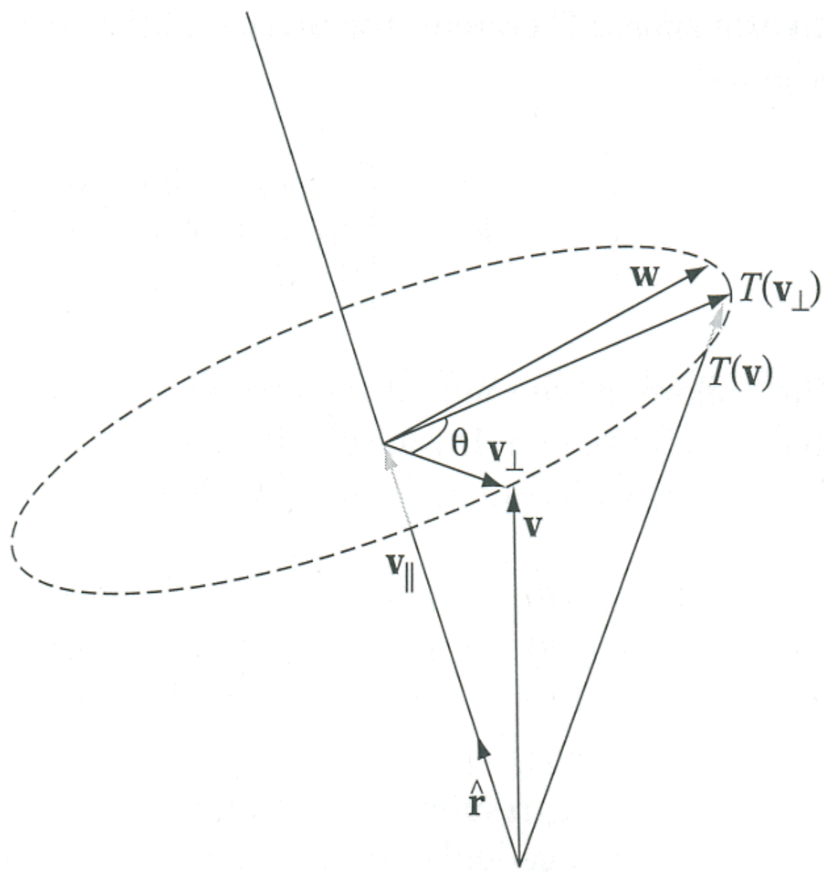

Figure 9.1: Rotation about a general axis through the origin, showing the axis of rotation and the plane of rotation (see [VB1])

Let T denote the rotation we are studying. We need to compute $T(\mathbf{v})$.

$$\begin{aligned} T(\mathbf{v}) &= T\left(\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}\right) \\ &= T\left(\mathbf{v}_{\parallel}\right) + T(\mathbf{v}_{\perp}) \end{aligned}$$

since T is a linear transformation. Also,

$$T\left(\mathbf{v}_{\parallel}\right) = \mathbf{v}_{\parallel}$$

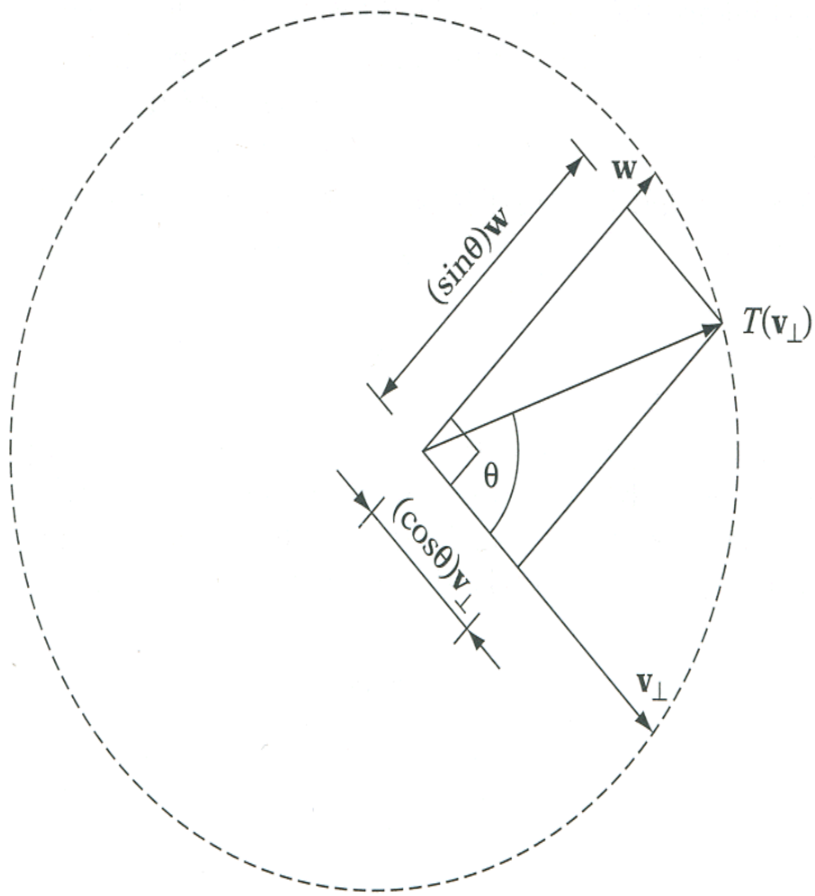

Figure 9.2: Rotation about a general axis through the origin, showing vectors on the plane of rotation (see [VB1])

since $\mathbf{v}_{\|}$ has the same direction as $\widehat{r}$ and we are rotating around an axis with direction vector $\widehat{r}$(see figure 9.1). Therefore,

$$T(\mathbf{v}) = \mathbf{v}_{\|} + T(\mathbf{v}_{\perp})$$

So, $T(\mathbf{v}_{\perp})$ is the only quantity we need to compute. For this, we create a two dimensional basis in the plane of rotation (see figure 9.1 and 9.2). We will use $\mathbf{v}_{\perp}$ as our first basis vector. For our second, we can use

$$\begin{aligned} \mathbf{w} &= \widehat{r} \times \mathbf{v}_{\perp} \\ &= \widehat{r} \times \mathbf{v} \end{aligned} \tag{9.2}$$

Looking at figure 9.2, we see that

$$\begin{aligned} T(\mathbf{v}_{\perp}) &= \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{w} \\ &= \cos\theta \mathbf{v}_{\perp} + \sin\theta (\widehat{r} \times \mathbf{v}) \end{aligned}$$

and therefore

$$\begin{aligned} T(\mathbf{v}) &= \mathbf{v}_{\|} + T(\mathbf{v}_{\perp}) \\ &= (\mathbf{v} \cdot \widehat{r})\widehat{r} + \cos\theta \mathbf{v}_{\perp} + \sin\theta (\widehat{r} \times \mathbf{v}) \\ &= (\mathbf{v} \cdot \widehat{r})\widehat{r} + \cos\theta [\mathbf{v} - (\mathbf{v} \cdot \widehat{r})\widehat{r}] + \sin\theta (\widehat{r} \times \mathbf{v}) \\ &= (\mathbf{v} \cdot \widehat{r})\widehat{r} + \cos\theta \mathbf{v} - \cos\theta (\mathbf{v} \cdot \widehat{r})\widehat{r} + \sin\theta (\widehat{r} \times \mathbf{v}) \\ &= (1 - \cos\theta)(\mathbf{v} \cdot \widehat{r})\widehat{r} + \cos\theta \mathbf{v} + \sin\theta (\widehat{r} \times \mathbf{v}) \end{aligned}$$

We would like to express this as a matrix transformation, in other words, we want to find the matrix R such that $T(\mathbf{v}) = R\mathbf{v}$. For this, we first establish some intermediary results.

Lemma 97 *If* $\mathbf{u} = (u_x, u_y, u_z)$ *and* $\mathbf{v} = (v_x, v_y, v_z)$ *then*

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \mathbf{v}$$

Proof.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix} \\ &= \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \mathbf{v} \end{aligned}$$

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Lemma 98 *Using the notation of the previous lemma, we have*

$$(\mathbf{u}\cdot\mathbf{v})\,\mathbf{u} = \begin{bmatrix} u_x^2 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 \end{bmatrix}\mathbf{v}$$

Proof.

$$\mathbf{u}\cdot\mathbf{v} = u_xv_x + u_yv_y + u_zv_z$$

So,

$$\begin{aligned} (\mathbf{u}\cdot\mathbf{v})\,\mathbf{u} &= (u_xv_x + u_yv_y + u_zv_z)\,\mathbf{u} \\ &= (u_xv_x + u_yv_y + u_zv_z)\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \\ &= \begin{bmatrix} u_x^2 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 \end{bmatrix}\mathbf{v} \end{aligned}$$

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We are now ready to write $T(\mathbf{v})$ as a matrix transformation.

$$\begin{aligned} T(\mathbf{v}) &= (1-\cos\theta)(\mathbf{v}\cdot\widehat{r})\,\widehat{r} + \cos\theta\mathbf{v} + \sin\theta\,(\widehat{r}\times\mathbf{v}) \\ &= (1-\cos\theta)\begin{bmatrix} u_x^2 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 \end{bmatrix}\mathbf{v} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cos\theta\mathbf{v} + \sin\theta\begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}\mathbf{v} \\ &= \left\{(1-\cos\theta)\begin{bmatrix} u_x^2 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cos\theta + \sin\theta\begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}\right\}\mathbf{v} \\ &= \begin{bmatrix} tu_x^2 + C & tu_xu_y - Su_z & tu_xu_z + Su_y \\ tu_xu_y + Su_z & tu_y^2 + C & tu_yu_z - Su_x \\ tu_xu_z - Su_y & tu_yu_z + Su_x & tu_z^2 + C \end{bmatrix}\mathbf{v} \end{aligned}$$

where

$$\begin{aligned} \widehat{r} &= (u_x, u_y, u_z) \\ C &= \cos\theta \\ S &= \sin\theta \\ t &= 1-\cos\theta \end{aligned}$$

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9.3 Assignment

1. Prove that the product of two orthogonal matrices is also an orthogonal matrix.

2. Prove equation 9.1
3. Prove equation 9.2
4. How would you implement rotation about an axis not going through the origin? First, consider the cases in which the axis of rotation is parallel to one of the coordinate axes. Then, consider the general case. In each case, derive the corresponding transformation matrix.

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